Chapter 4 Concepts from Geometry

An Introduction to Optimization

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Line Segments

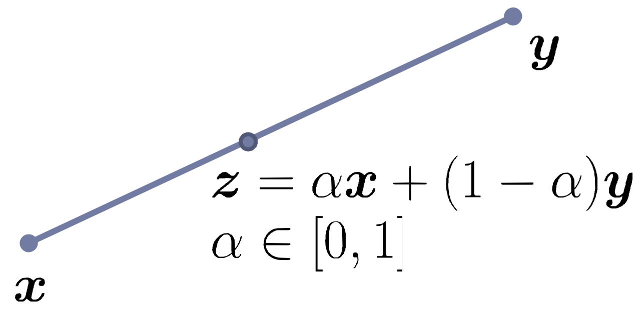
- The line segment between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in R^n is the set of points on the straight line joining points $\boldsymbol{x}$ and $\boldsymbol{y}$. If $\boldsymbol{z}$ lies on the line segment, then

$$\boldsymbol{z} - \boldsymbol{y} = \alpha(\boldsymbol{x} - \boldsymbol{y})$$

$$\boldsymbol{z} = \alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}$$

- Hence, the line segment between $\boldsymbol{x}$ and $\boldsymbol{y}$ can be represented as

$$\{\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} : \alpha \in [0,1]\}$$



Hyperplanes and Linear Varieties

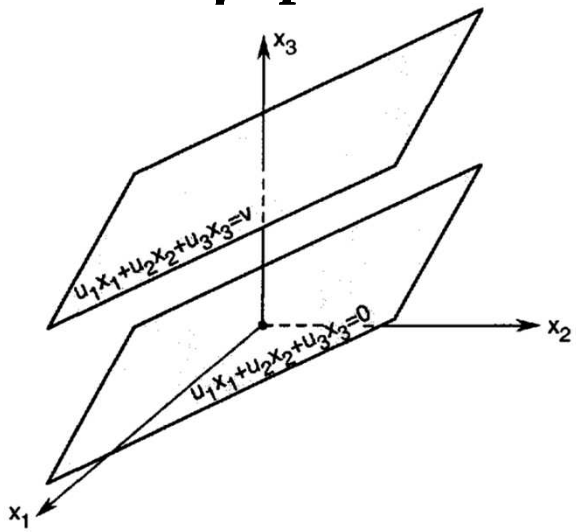
- Let $u_1, u_2, ..., u_n, v \in R$ where at least one of the u_i is nonzero. The set of all points $\boldsymbol{x} = [x_1, x_2, ..., x_n]^T$ that satisfy the linear equation $u_1x_1 + u_2x_2 + \cdots + u_nx_n = v$ is called a ***hyperplane*** of the space R^n.
- We may describe the hyperplane by

$$\{\boldsymbol{x} \in R^n : \boldsymbol{u}^T\boldsymbol{x} = v\} \qquad \boldsymbol{u} = [u_1, u_2, ..., u_n]^T$$

- A hyperplane is not necessarily a subspace of R^n since, in general, it does not contain the origin.
- For $n = 2$, the hyperplane has the form $u_1x_1 + u_2x_2 = v$, which is a straight line. In R^3, hyperplanes are ordinary planes.

Hyperplanes and Linear Varieties

- By translating a hyperplane so that it contains the origin of R^n, it becomes a subspace of R^n. Because the dimension of this subspace is n-1, we say that the hyperplane has dimension n-1.
- The hyperplane $H = \{\boldsymbol{x} : u_1x_1 + \cdots + u_nx_n = v\}$ divides R^n into two ***half-spaces***. One satisfies the inequality $u_1x_1 + \cdots + u_nx_n \geq v$ denoted by $H_+ = \{\boldsymbol{x} \in R^n : \boldsymbol{u}^T\boldsymbol{x} \geq v\}$, and the another one satisfies $u_1x_1 + \cdots + u_nx_n \leq v$, denoted by $H_- = \{\boldsymbol{x} \in R^n : \boldsymbol{u}^T\boldsymbol{x} \leq v\}$
- The half-spaces H_+ and H_- are called ***positive half-space*** and ***negative half-space***, respectively.

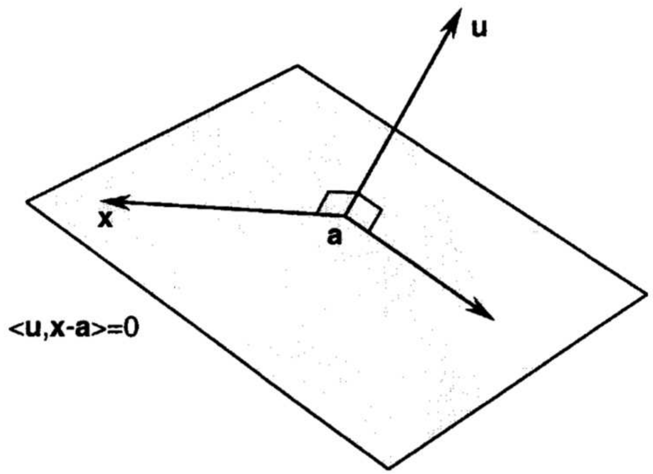


Hyperplanes and Linear Varieties

- Let $\boldsymbol{a} = [a_1, a_2, ..., a_n]^T$ be an arbitrary point of the hyperplane *H*. Thus, $\boldsymbol{u}^T\boldsymbol{a} - v = 0$. We can write

$$\boldsymbol{u}^T\boldsymbol{x} - v = \boldsymbol{u}^T\boldsymbol{x} - v - (\boldsymbol{u}^T\boldsymbol{a} - v)$$
$$= \boldsymbol{u}^T(\boldsymbol{x} - \boldsymbol{a})$$
$$= u_1(x_1 - a_1) + u_2(x_2 - a_2) + \cdots + u_n(x_n - a_n) = 0$$

- The hyperplane *H* consists of the points $\boldsymbol{x}$ for which $\langle \boldsymbol{u}, \boldsymbol{x} - \boldsymbol{a} \rangle = 0$ In other words, the hyperplane *H* consists of the points $\boldsymbol{x}$ for which the vectors $\boldsymbol{u}$ and $\boldsymbol{x} - \boldsymbol{a}$ are orthogonal. The vector $\boldsymbol{u}$ is the ***normal*** to the hyperplane *H*.
- The set H_+ consists of those points $\boldsymbol{x}$ for which $\langle \boldsymbol{u}, \boldsymbol{x} - \boldsymbol{a} \rangle \geq 0$ and H_- consists of those points $\boldsymbol{x}$ for which $\langle \boldsymbol{u}, \boldsymbol{x} - \boldsymbol{a} \rangle \leq 0$



Hyperplanes and Linear Varieties

- A ***linear variety*** is a set of the form
$$\{\boldsymbol{x} \in R^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$$
for some matrix $\boldsymbol{A} \in R^{m \times n}$ and vector $\boldsymbol{b} \in R^m$
- If $\dim \mathcal{N}(\boldsymbol{A}) = r$, we say that the linear variety has dimension *r*. A linear variety is a subspace if and only if $\boldsymbol{b} = \mathbf{0}$. If $\boldsymbol{A} = \boldsymbol{O}$, the linear variety is R^n.
- If the dimension of the linear variety is less than *n*, then it is the intersection of a finite number of hyperplanes.

Linear Varieties

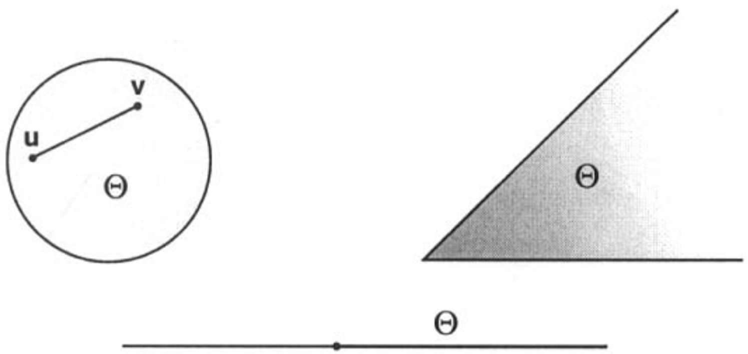
- A linear variety is also called a *linear manifold* or *flat*.
- A linear variety can be described by a system of linear equations. For example, a line in two-dimensional space

$$3x + 5y = 8$$

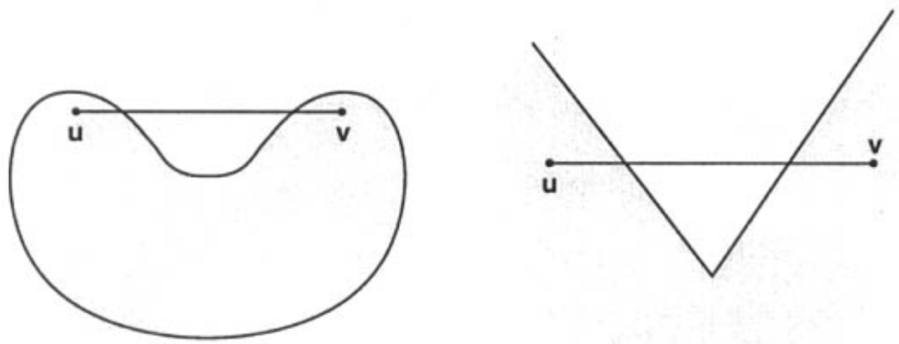
- In three-dimensional space, a single linear equation involving x, y, and z defines a plane, while a pair of linear equations can be used to describe a line.
- In general, a linear equation in n variables describes a hyperplane, and a system of linear equations describes the intersection of those hyperplanes.
- Assuming the equations are consistent and linearly independent, a system of k equations describes a flat of dimension n-k.

Convex Sets

- Recall that the line segment between two points $\boldsymbol{u}, \boldsymbol{v} \in R^n$ is the set $\{\boldsymbol{w} \in R^n : \boldsymbol{w} = \alpha \boldsymbol{u} + (1-\alpha)\boldsymbol{v}, \alpha \in [0,1]\}$. A point $\boldsymbol{w} = \alpha \boldsymbol{u} + (1-\alpha)\boldsymbol{v}$ (where $\alpha \in [0,1]$) is called a ***convex combination*** of the points $\boldsymbol{u}$ and $\boldsymbol{v}$
- A set $\Theta \subset R^n$ is ***convex*** if for all $\boldsymbol{u}, \boldsymbol{v} \in \Theta$, the line segment between $\boldsymbol{u}$ and $\boldsymbol{v}$ is in Θ



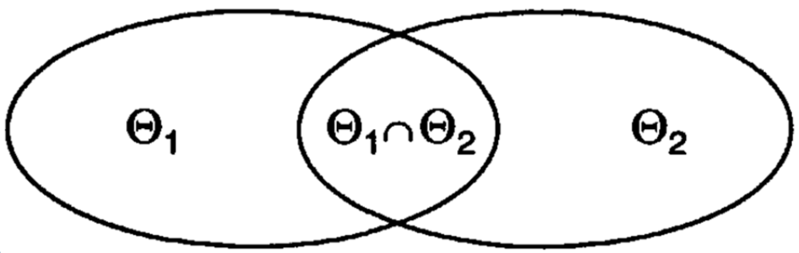
Convex sets



Sets that are not convex

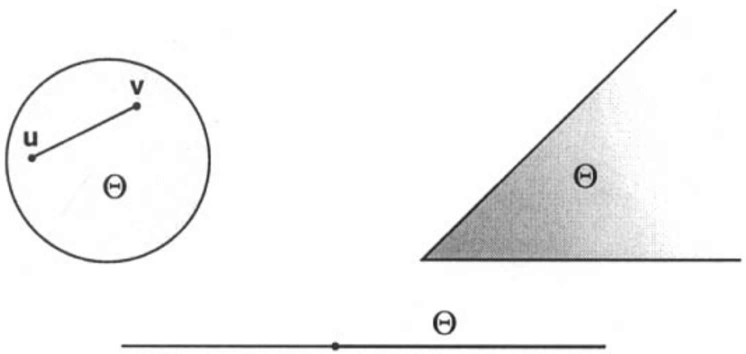
Convex Sets

- Examples of convex sets
  - The empty set; a set consisting of a single point; a line or a line segment; a subspace; a hyperplane; a linear variety; a half-space; R^n
- Theorem 4.1: Convex subsets of R^n have the following properties:
  - If Θ is a convex set and β is a real number, then the set $\beta\Theta = \{\boldsymbol{x} : \boldsymbol{x} = \beta\boldsymbol{v}, \boldsymbol{v} \in \Theta\}$ is also convex.
  - If Θ_1 and Θ_2 are convex sets, then the set $\Theta_1 + \Theta_2 = \{\boldsymbol{x} : \boldsymbol{x} = \boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_1 \in \Theta_1, \boldsymbol{v}_2 \in \Theta_2\}$ is also convex.
  - The intersection of any collection of convex sets is convex.



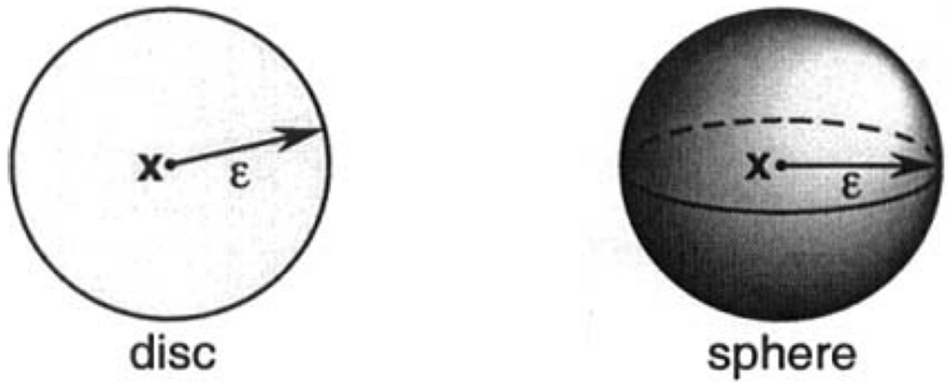
Convex Sets

- A point $\boldsymbol{x}$ in a convex set Θ is said to be an ***extreme point*** of Θ if there are no two distinct points $\boldsymbol{u}$ and $\boldsymbol{v}$ in Θ such that $\boldsymbol{x} = \alpha\boldsymbol{u} + (1-\alpha)\boldsymbol{v}, \alpha \in [0,1]$
- For example, any point on the boundary of the disk is an extreme point, the vertex (corner) of the set on the right is an extreme point, and the endpoint of the half-line is also an extreme point.



Neighborhoods

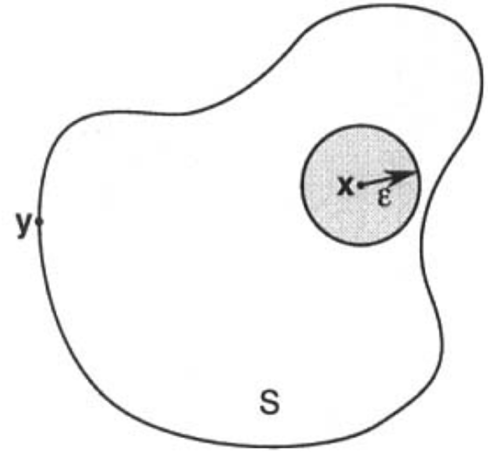
- A ***neighborhood*** of a point $\boldsymbol{x} \in R^n$ is the set $\{\boldsymbol{y} \in R^n : \|\boldsymbol{y} - \boldsymbol{x}\| < \epsilon\}$ where ϵ is some positive number. The neighborhood is also called a ***ball*** with radius ϵ and center $\boldsymbol{x}$.
- In the plane R^2, a neighborhood of $\boldsymbol{x} = [x_1, x_2]^T$ consists of all the points inside a disk centered at $\boldsymbol{x}$. In R^3, a neighborhood of $\boldsymbol{x} = [x_1, x_2, x_3]^T$ consists of all the points inside a sphere centered at $\boldsymbol{x}$.



A neighbourhood of a point is a set containing the point where you can move that point some amount without leaving the set.

Neighborhoods

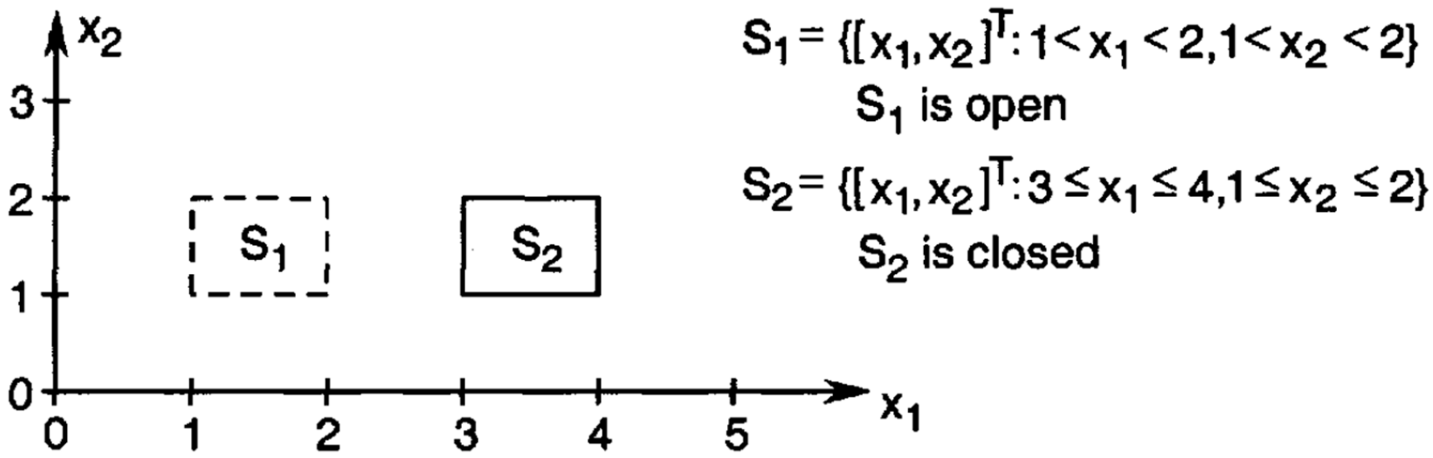
- A point $x \in S$ is said to be an ***interior point*** of the set *S* if the set *S* contains some neighborhood of x; that is, if all points within some neighborhood of x are also in *S*. The set of all the interior points of *S* is called the ***interior*** of *S*.



- A point x is said to be a ***boundary point*** of the set *S* if every neighborhood of x contains a point in *S* and a point not in *S*. The set of all boundary points of *S* is said the ***boundary*** of *S*.

Neighborhoods

- A set S is said to be ***open*** if it contains a neighborhood of each of its points; that is, if each of its points is an interior point, or equivalently, if S contains no boundary points.
- A set S is said to be ***closed*** if it contains its boundary.



Neighborhoods

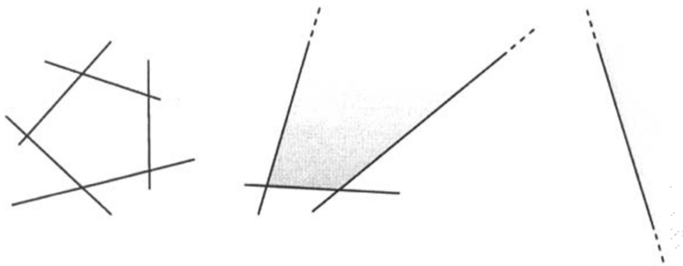
- A set that is contained in a ball of finite radius is said to be ***bounded***. A set is ***compact*** if it is both closed and bounded. Compact sets are important in optimization problems.
- **Theorem 4.2: Theorem of Weierstrass**: Let $f : \Omega \to R$ be a continuous function, where $\Omega \subset R^n$ is a compact set. Then, there exists a point $\boldsymbol{x}_0 \in \Omega$ such that $f(\boldsymbol{x}_0) \leq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$. In other words, f achieves its minimum on Ω.

Polytopes and Polyhedra

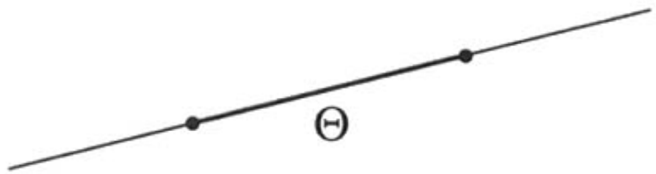
- Let Θ be a convex set, and suppose that y is a boundary point of Θ A hyperplane passing through y is called a ***hyperplane of support*** (or ***supporting hyperplane***) of the set Θ if the entire set Θ lies completely in one of the two half-spaces into which this hyperplane divides the space R^n.
- Recall that the intersection of any number of convex sets is convex. Because every half-space H_+ or H_- is convex in R^n, the intersection of any number of half-spaces is a convex set.

Polytopes and Polyhedra

- A set that can be expressed as the intersection of a finite number of half-spaces is called a ***convex polytope*** (凸多胞形).
- A nonempty bounded polytope is called a ***polyhedron*** (多面體).
- For every convex polyhedron $\Theta \subset R^n$, there exists a nonnegative integer $k \le n$ such that Θ is contained in a linear variety of dimension k, but is not entirely contained in any $(k-1)$-dimensional linear variety of R^n.

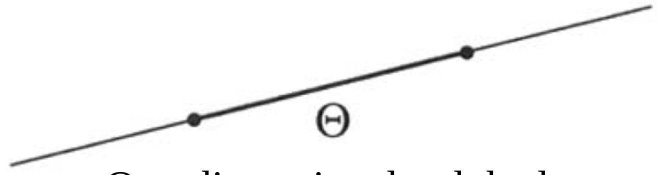


Polytopes



One-dimensional polyhedron

Polytopes and Polyhedra



One-dimensional polyhedron

- There exists only one k-dimensional linear variety containing Θ, called the ***carrier*** of the polyhedron Θ, and k is called the ***dimension*** of Θ.
- For example, a zero-dimensional polyhedron is a point of R^n, and its carrier is itself. A one-dimensional polyhedron is a segment, and its carrier is the straight line on which it lies.
- The boundary of any k-dimensional polyhedron, $k > 0$, consists of a finite number of $(k-1)$-dimensional polyhedra. For example, the boundary of a one-dimensional polyhedron consists of two points that are the endpoints of the segment.

Polytopes and Polyhedra

- The $(k-1)$-dimensional polyhedra forming the boundary of a k-dimensional polyhedron are called the ***faces*** of the polyhedron. Each of these faces has, in turn, $(k-2)$-dimensional faces.
- We also consider each of these $(k-2)$-dimensional faces to be faces of the original k-dimensional polyhedron. Thus, every k-dimensional polyhedron has faces of dimensions $k-1, k-2, ..., 1, 0$
- A zero-dimensional face of a polyhedron is called a ***vertex***, and a one-dimensional face is called an ***edge***.